Notes on Graph $C^*$-Algebras and Dynamical Systems

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1 $C^*$-algebras

Abstract Definition

§1.1 Definition.
A $C^*$-algebra $A$ is a (complex) Banach algebra together with an involution $(\cdot)^*$ satisfying the following equations.

\[ t^{**} = t \quad (t + u)^* = t^* + u^* \quad (\lambda t)^* = \bar{\lambda} t^* \quad (tu)^* = u^* t^* \]

\[ \|t^*\| = \|t\| \quad \|t^* t\| = \|t\|^2 \quad (\lambda \in \mathbb{C}, t, u \in A) \]

The last equation is called the $C^*$-identity.

§1.2 If $A$ is non-unital, we can define its unitarisation $A^+$ as the algebra $A \times \mathbb{C}$ where scalar multiplications and sums are defined by $\lambda(a, \mu) + (b, \nu) = (\lambda a + b, \lambda \mu + \nu)$, and multiplication is defined as $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda \mu)$. We can define on $A^+$ the norm $\|(a, \lambda)\| = \|a\| + |\lambda|$; $A^+$ is complete w.r.t. this norm, and the involution
\((a, \lambda)^* = (a^*, \bar{\lambda})\) satisfies all the equations shown above. Thus \(A^+\) is a \(C^*\)-algebra, it is unital (the unit is \((0, 1)\)), and there is a natural embedding (a \(*\)-morphism) of \(A\) into \(A^+\) defined by \(a \mapsto (a, 0)\).

§1.3 Definition.

For any element \(a\) in a unital \(C^*\)-algebra \(A\), we define its spectrum

\[
\text{Spec}_A(a) = \{ \lambda \in \mathbb{C} \mid a - \lambda \cdot 1 \text{ non-invertible} \}
\]

When \(A\) is non-unital, the spectrum of an element \(a\) is defined as the spectrum of its image in the unitarisation \(A^+\) of \(A\).

§1.4 Lemma.

If \(\phi: A \to B\) is a \(*\)-morphism, then for all \(a \in A\), \(\text{Spec}_B(\phi(a)) \subseteq \text{Spec}_A(a)\).

§1.5 Proposition.

Let \(A\) be a \(C^*\)-algebra and let \(a\) be an element of \(A\). Then \(\text{Spec}_A(a)\) is a non-empty closed and bounded subset of the field of complex numbers.

§1.6 This last lemma will be of use later on. We will next state a theorem providing a characterisation of the spectrum of elements of a \(C^*\)-algebra. Before that, however, we fix some terminology. Among the elements of a \(C^*\)-algebra \(A\), we distinguish:

- **Projections**: elements \(p \in A\) such that \(p = p^2 = p^*\);
- **Unitaries**: elements \(u \in A\) such that \(uu^* = u^*u = 1\);
- **Partial Isometries**: elements \(u \in A\) such that \(uu^*\) is a projection;
- **Normal Elements**: elements \(n \in A\) such that \(nn^* = n^*n\);
- **Hermitian Elements**: elements \(h \in A\) such that \(h = h^*\);
- **Positive Elements**: hermitian elements \(a \in A\) with \(\text{Spec}_A(a) \subseteq \mathbb{R}_{\geq 0}\);

The following standard characterisations of positive elements and partial isometries is very useful.

§1.7 Proposition.

The set \(A^+\) of positive elements in a \(C^*\)-algebra \(A\) can be characterised as follows:

\[
A^+ = \{ b^*b \mid b \in A \}
\]

§1.8 Proposition.

The followings are equivalent:

- \(uu^*\) is a projection;
- \(u^*u\) is a projection;
- \(uu^* = u^*u\);
- \(u^*uu^* = u^*\);

**Proof.** Suppose \(uu^* u = u\). Then \(u^*uu^* = (uu^*)^* u = u^*\). This shows that 3 and 4 are equivalent.

Moreover, if \(uu^* u = u\), then \((uu^*)^2 = uu^* = (uu^*)^*\), thus \(uu^*\) is a projection. Therefore 3 implies 1. Similarly, 4 implies 2.
We now only need to show that if $u^* u$ is a projection, then $uu^* u = u$. Let us write $v = uu^* u - u$ and compute:

$$v^* v = (u^* uu^* - u^*)(uu^* u - u)$$
$$= u^* uu^* uu^* u - u^* uu^* u - u^* uu^* + u^* u$$
$$= (u^* u)^3 - 2(u^* u)^2 + u^* u$$
$$= 0$$

From $v^* v = 0$, we deduce $v = 0$ since $\|v\|^2 = \|v^* v\| = 0$, and therefore $uu^* u = u$, which shows that 1 implies 3. One shows similarly that 2 implies 4, ending the proof. 

\[\]
§1.16 Let $a \in A$ be an operator. We define $\phi(a)$ from $A/N_\tau$ to $A/N_\tau$ by:

$$\phi(a)b = ab$$

One can then show that $\|\phi(a)\| \leq \|a\|$ and we denote by $\phi_\tau$ the unique extension of $\phi$ to $\mathfrak{h}_\tau$. We can then show that $\phi_\tau$ is a $\ast$-morphism.

§1.17 We have therefore defined, for every positive linear form $\tau$, a representation $(\mathfrak{h}_\tau, \phi_\tau)$. The universal representation of $A$ is then defined as the direct sum of all such representations:

$$(\mathcal{H}_U, \phi_U) = \bigoplus \tau (\mathfrak{h}_\tau, \phi_\tau)$$

§1.18 Theorem.

Let $A$ be a $C^*$-algebra. Then $A$ has a faithful representation. In particular, the universal representation of $A$ is faithful.

Gelfand Theorem

Now, before going on with more specific topics, let us quickly mention the celebrated Gelfand theorem. This theorem is about commutative $C^*$-algebras. Defining commutative $C^*$-algebras is not very difficult. Chosing a compact Hausdorff space $X$, one can consider the set of all continuous functions from $X$ to the field of complex numbers $\mathcal{C}(X)$. This set is a commutative algebra when considered with pointwise scalar multiplication, product and multiplication:

$$(\lambda f)(x) = \lambda \times f(x) \quad (f + g)(x) = f(x) + g(x) \quad (f \times g)(x) = f(x) \times g(x)$$

It is moreover a $C^*$-algebra when considered with the involution $f^\ast(x) = \overline{f(x)}$, where $\overline{}$ denotes complex conjugation, and the norm $\|f\| = \sup_X |f(x)|$. Let us notice among other things that the spectrum of $f \in \mathcal{C}(X)$ is exactly the set $\{f(X)\}$, that hermitians are real-valued functions and positive elements are positive-real-valued functions.

Similarly, the set $\mathcal{C}_0(X_0)$ of continuous functions that vanish at infinity on a locally compact Hausdorff space $X_0$ is also a commutative $C^*$-algebra. While $\mathcal{C}(X)$ is a unital $C^*$-algebra, $\mathcal{C}_0(X_0)$ is not. In fact, the existence of the unit is the algebraic counterpart of the fact that the underlying space is compact.

Gelfand’s theorem mainly states that all commutative $C^*$-algebras are of this form, i.e. (in the unital case) given a unital commutative $C^*$-algebra $A$, one can build a compact Hausdorff space $\Omega(A)$ such that $A \cong \mathcal{C}(\Omega(A))$. This theorem can in fact be understood more generally in category-theoretic terms as follows: the category of compact Hausdorff spaces with continuous functions is equivalent to (the opposite of) the category of unital commutative $C^*$-algebras. The non-unital case morally extends this by replacing the compactness condition to allow for locally compact spaces; some restrictions should however be considered on morphisms. What will be of use in the following is the major corollary of Gelfand’s theorem, namely the existence of the continuous functional calculus.

§1.19 Theorem.

Let $A$ be a $C^*$-algebra. The commutative $C^*$-algebra generated by an element $a \in A$ such that $aa^\ast = a^\ast a$ (such elements are usually called normal) is isomorphic to the algebra of functions over $\text{Spec}_A(a)$.

The consequence of this fact is the existence of the functional calculus: if $g$ is a continuous function from $\text{Spec}_A(a)$ to $\mathbb{C}$, then one can define the function $g(a)$ as
follows. The previous theorem states that there exists an injective morphism $\psi$ from $\mathcal{C}(\text{Spec}_A(a))$ to $A$ such that $\psi(i) = a$ ($i$ denotes the inclusion of $\text{Spec}_A(a)$ into $C$). Then we define $g(a)$ as the operator $\psi(g)$. This can be used, for instance, to define the square root of arbitrary positive operator $p$, and of course the resulting operator satisfies the properties of the square root: it is a positive operator $q$ such that $q^2 = p$.

2 Universal C*-algebras

Definitions


§2.1 Definition (Generators and Relations Presentation).
A generator and relations presentation $(G, \mathcal{R})$ is a pair of a set $G = \{x_\alpha\}$ of generators and a set $\mathcal{R} = \{R_\beta\}$ of relations, i.e. $R_\beta$ is of the form

$$\|p(x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}, x_{*\alpha_1}, x_{*\alpha_2}, \ldots, x_{*\alpha_n})\| \leqslant \eta,$$

where $p$ is a complex polynomial in $2n$ non-commuting variables, and $\eta > 0$.

§2.2 Definition (Representation).
A representation of such a pair $(G, \mathcal{R})$ is a pair $(H, \{a_\alpha\})$ of a Hilbert space $H$ and a family of operator $\{a_\alpha\}$ on $H$ such that for all relation $R_\beta$, the inequality $R(a_{\alpha_1}/x_{\alpha_1})$ is satisfied. Such a representation extends uniquely to a *-homomorphism from the free *-algebra $\mathcal{F}(G)$ generated by $G$ into $\mathcal{L}(H)$.

§2.3 Definition (Admissible Presentations).
A generator and relations presentation $(G, \mathcal{R})$ is admissible if:

• it has a representation;

• for all family $(H, \{y_\gamma\})$ of representations, $\bigoplus_\gamma y_\gamma \in \mathcal{L}(\bigoplus_\gamma H)$.

§2.4 The first condition should be understood as the compatibility of the presentation with the C*-algebra axioms. A counterexample to this condition is $(\{x\}, \{\|xx^*\| = 0, \|x\| = 1\})$ which is breaking the C*-identity $\|aa^*\| = \|a\|^2$.

§2.5 The second condition insures that the relations place a bound on the norm of $x_\alpha$. A typical example of a non-admissible presentation breaking this condition would be the pair $(\{x\}, \{\cdot, \cdot\})$. Indeed, any pair $(H, \{a\})$ of a Hilbert space and an operator $a$ on $H$ is a representation of this. Taking the family of representations $(\{H, \{n.1\}\})_{n \in \mathbb{N}}$ provides a counterexample to the second condition.

§2.6 Lemma.
If a generator and relations presentation $(G, \mathcal{R})$ is admissible, then for any $z \in \mathcal{F}(G)$, the set

$$\{\|\rho(z)\| \mid \rho \text{ a representation of } (G, \mathcal{R})\}$$

has a supremum, denoted by $|||z|||$, which defines a C* -seminorm.

§2.7 Definition.
The completion of $\mathcal{F}(G)/\{z \mid |||z||| = 0\}$ is called the universal C*-algebra of the presentation $(G, \mathcal{R})$ and is denoted by $C^*(G, \mathcal{R})$.

§2.8 Proposition.
Any representation of $(G, \mathcal{R})$ extends uniquely to a representation of $C^*(G, \mathcal{R})$. Conversely, any representation of $C^*(G, \mathcal{R})$ defines a representation of $(G, \mathcal{R})$. 5
Examples

§2.9 Let \(A\) be a \(C^\ast\)-algebra. We write \(G = A\) and \(\mathcal{R}\) the set of all *-algebraic relations in \(A\). Then \(C^\ast(G, \mathcal{R}) \cong A\).

§2.10 In the previous example, let us take \(G = A_0\) a dense *-subring of \(A\) which is an algebra over a dense subfield of \(\mathbb{C}\). Then, take \(\mathcal{R}\) as the set of all *-algebraic relations in \(A_0\) to which we add (1) scalar multiples relations between elements of \(A_0\) and (2) a relation \(\|x\| \leq \|x\|_A\) for each \(x \in A_0\). Then \(C^\ast(G, \mathcal{R}) \cong A\). In particular, if \(A\) is separable then it is the universal algebra for a countable generators and relations presentation.

§2.11 Let \(\mathcal{G}\) be a discrete group. Then \(C^\ast(G, \mathcal{R}) \cong C^\ast(\mathcal{G})\) for \(G = \mathcal{G}\) and \(\mathcal{R}\) is the set of all relations \(x^* x = x x^* = 1\) and \(x y = z\) for all \(x, y, z \in \mathcal{G}\) satisfying \(x y = z\).

§2.12 Let \(G = \{x\}\) and \(\mathcal{R} = \{x x^* = x^* x = 1\}\). Then \(C^\ast(G, \mathcal{R}) \cong \mathcal{C}(S_1) \cong C^\ast(\mathbb{Z})\).

- First, let us remark that any representation of \((G, \mathcal{R})\) is given by a Hilbert space together with a unitary operator \(u\). It is known that any unitary operator \(u\) is such that Spec\((u) = S_1\). From the universal property we can deduce that the universal representation \((\mathcal{H}, U)\) should be such that Spec\((U) = S_1\). If it were not the case, then another representation \((\mathcal{K}, v)\) of \((G, \mathcal{R})\) could not extend to a representation of \(C^\ast(G, \mathcal{R})\) since *-morphisms \(\phi\) necessarily satisfy Spec\((\phi(a)) \subset\text{Spec}(a)\). Then, from the continuous functional calculus, we know that the \(C^\ast\)-algebra generated by \(U\) is isomorphic to the algebra of continuous function over its spectrum, i.e. \(C^\ast(G, \mathcal{R}) \cong \mathcal{C}(S_1)\).

- Second, let us notice that \(C^\ast(\mathbb{Z})\) is a commutative \(C^\ast\)-algebra. Moreover, it is by definition generated by the unitaries \(u_i\) for \(i \in \mathbb{Z}\), which satisfy \(u_i u_j = u_{i+j}\). Consequently, it is the algebra generated by \(u_1\), taking into account that \(u_1^1 = u_1 = u^\ast - 1\). This shows that \(C^\ast(\mathbb{Z}) \cong \mathcal{R} = \{x x^* = x^* x = 1\}\).

§2.13 Let \(G = \{x\}\) and \(\mathcal{R} = \{x x^* = 1\}\). Then \(C^\ast(G, \mathcal{R}) \cong C^\ast(\mathcal{G})\) where \(u\) is the unilateral shift (also called the Toeplitz algebra). (see Wold’s decomposition).

§2.14 Let \(G = \{x, y\}\) and \(\mathcal{R} = \{x x^* = x^* x = y y^* = y^* y = 1, xy = e^{2\pi i/a} y x\}\). Then \(C^\ast(G, \mathcal{R})\) is the irrational rotation algebra \(\mathcal{A}_a\).

§2.15 Let \(G = \{x_i, j \mid 1 \leq i, j \leq n\}\) and \(\mathcal{R} = \{x_i, j = x_j, i x, i j x, k, j = \delta, j, k x_i, j\}\). Then \(C^\ast(G, \mathcal{R})\) is the \(n \times n\) matrix algebra \(\mathcal{M}_n(\mathbb{C})\).

- To prove this, let us consider a representation \((\mathcal{H}, \{a_{ij}\})\) of \((G, \mathcal{R})\). We can define a map from the matrix algebra \(\mathcal{M}_n(\mathbb{C})\) to \(\mathcal{L}(\mathcal{H})\) as follows: each matrix \(\{a_{ij}\}\) is mapped to the element \(\sum_{i, j} a_{i, j} a_{i, j}\). It is not difficult to see that this is a *-morphism. i.e. we just proved that any representation of \((G, \mathcal{R})\) extends to a representation of \(\mathcal{M}_n(\mathbb{C})\). We now need to prove the unicity of such an extension, i.e. we need to prove that \(\mathcal{M}_n(\mathbb{C})\) is the “smallest” algebra having this property. For this, consider that you have two extensions \(\phi, \psi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{L}(\mathcal{H})\) of the representation \((\mathcal{H}, \{a_{ij}\})\). They coincide on the natural basis \(E_{i, j}\) of matrices because they extend the representation \((\mathcal{H}, \{a_{ij}\})\), and they are therefore equal.

§2.16 Let \(G = \{x, i, j \mid l, j \in \mathbb{N}\}\) and \(\mathcal{R} = \{x_i, j = x_j, i x_i, j x_k, l = \delta, j, k x_i, l\}\). Then \(C^\ast(G, \mathcal{R})\) is the \(C^\ast\)-algebra of compact operators \(\mathcal{K}(\mathcal{H})\).

- One can try to extend a representation \((\mathcal{H}, \{a_{ij}\})\) of \((G, \mathcal{R})\) to a representation of \(\mathcal{L}(\mathcal{H})\). This does not define a *-morphism as in the example above. In fact
the definition considered above does not make sense because of divergence issues: if \( (\mathcal{H}, \{a_{i,j}\}) \) is \((C, 1)\), then the image of an operator \( \pi \in \mathcal{L}(\mathcal{H}) \) is the sum \( \sum_{i,j} a_{i,j} \) where \( a_{i,j} = (ue_i, e_j) \) for a basis \( (e_i) \) of \( \mathcal{H} \), and this sum is divergent if one takes, for instance, the operator defined by \( A e_i = \frac{1}{i} e_i \). Therefore \( \mathcal{L}(\mathcal{H}) \) is "too big" to be the universal algebra we are looking for. On the other hand, the representation \( (\mathcal{H}, \{a_{i,j}\}) \) extends (but not uniquely!) to representations of matrix algebras. We are therefore looking for some algebra containing all matrix algebras. The good candidate is \( \mathcal{K}(\mathcal{H}) \), the algebra of compact operators. The following theorem can be used to prove that it is in fact the algebra we are looking for: \( T \) is a compact operator if and only if there exists a sequence \( T_n \) of finite rank operators such that \( \| T - T_n \| \to 0 \). This implies unicity of the extended representation because this extension is uniquely defined from the representation of finite rank operators, whose representation is uniquely determined by the representation of \( (\mathcal{H}, \{a_{i,j}\}) \).

3 Graphs C*-algebras

Definitions

§3.1 Notations. We define graphs as families \( F = (E, V, s, t) \) where \( E, V \) are (countable) sets of edges and vertices respectively, and \( s, t : E \to V \) are functions called source and target maps, respectively. A path of length \( n \) in a graph \( F \) is a pair \( \pi = (\vec{v}, \vec{e}) \) of a sequence \( \vec{v} = v_0 \ldots v_n \) of vertices and a sequence \( \vec{e} = e_1 \ldots e_n \) of edges such that for all \( i = 1, \ldots, n \), \( s(e_i) = v_{i-1} \) and \( t(e_i) = v_i \).

Source and target maps are naturally extended to paths as follows: \( s(\pi) = v_0 \) and \( t(\pi) = v_n \). Given two paths \( \pi = (\vec{v}, \vec{e}) \) and \( \pi' = (\vec{u}, \vec{f}) \) such that \( s(\pi') = s(\pi) \), we can form the path \( \pi \cdot \pi' = (\vec{v} \cup \vec{u}, \vec{e} \cup \vec{f}) \) of length \( n + m \) defined by \( \vec{v} = v_0 \ldots v_n u_1 \ldots u_m \) and \( \vec{f} = e_1 \ldots e_n f_1 \ldots f_m \). We will in the following identify edges with paths of length 1 and therefore use the notation \( \pi \cdot e \) for the path obtained as the concatenation of the path \( e \) to the right of the path \( \pi \) (when it is defined). In the same way, vertices are identified with paths of length 0, and we will allow for the notation \( \pi \cdot v \) for the concatenation of \( \pi \) with the path \((v, e)\) where \( e \) is the empty sequence.

§3.2 Definition. Let \( F \) be a graph. A Cuntz-Krieger \( F \)-family \( \{S, P\} \) on a Hilbert space \( \mathcal{H} \) consists of mutually orthogonal projections \( \{P_v \mid v \in V^F\} \) and partial isometries \( \{S_e \mid e \in E^F\} \) satisfying:

- (CK1) for all \( e, f \in E^F \) we have \( S_e^* S_f = \delta_{e,f} P_{s(e)} \);
- (CK2) for all \( v \in V^F \) which is a target of a non-zero finite number of edges, \( P_v = \sum_{e \in E^F, t(e) = v} S_e S_e^* \);
- (CK2') for all \( e \in E^F \) we have \( S_e S_e^* \leq P_{t(e)} \).

§3.3 Remark. In case of \( G \) being a graph satisfying that for all vertex \( v \), the number of edges having \( v \) as target is finite, the last condition (CK2') is superfluous. Indeed, if \( v \) satisfies the condition of condition (CK2), then (CK2') is a direct consequence of the equality \( P_v = \sum_{e \in E^F, t(e) = v} S_e S_e^* \). The last condition (CK2') is thus introduced for dealing with arbitrary graphs, and is necessary only when a vertex is target of an infinite number of edges. In that case, condition (CK2) does not apply, as it would
§3.5 Let us consider a direct construction of a non-degenerate Cuntz-Krieger family. In general, an edge $e$ in a graph will give rise to a partial isometry $S_e$ such that $S_eS_e^* = P_{s(e)}$ and $S_{t(e)}S_e \ll P_{t(e)}$, i.e. $S_e$ has as source a subprojection of $P_{t(e)}$ and as target the projection $P_{s(e)}$. We chose the converse convention which seemed more natural from a graph-theoretic point of view, although it has its own disadvantages (for instance, composition of partial isometries reverses the order of the edges in a path). In any case, the reader should be fully aware that this is only a choice of design which does not affect the theory and results.

§3.6 The preceding construction defines a non-degenerate Cuntz-Krieger family for any graph $F$. However, this representation need not be separable, as the space $\ell^2(\partial E)$ is not separable in general. Indeed, the graph with on vertex and two edges has an uncountable loop space $E^0$ (there is an injection of all infinite binary words into $E^0$), hence $\partial E$ is not countable which implies that $\ell^2(\partial E)$ is not separable.

We will now see that there is in fact a non-degenerate Cuntz-Krieger family for any graph $F$ on a separable Hilbert space. For any finite path $\mu = (\vec{v}, \vec{e})$, we will denote by $S_\mu$ the operator $P_{v_0}S_{e_n}P_{v_{n-1}}S_{e_{n-1}} \ldots P_{v_1}S_{e_1}P_{v_0}$. Notice that from the Cuntz-Krieger family conditions:

- when $\mu$ is of length at least 1, $S_\mu = S_{e_n} \ldots S_{e_1}$;
- when $\mu = (v, e)$ is of length 0, $S_\mu = P_v$;
- when $\mu, v$ are paths with $s(v) = t(\mu)$, $S_vS_\mu = S_{\mu v}$.

§3.7 Definition.
A graph is fibered if all vertices are the target of at least one edge.

§3.8 Lemma.
Let $\{S, P\}$ be a non-degenerate Cuntz-Krieger family for a graph $F$. Then:
1. For any finite path $\mu$, the operator $S_\mu$ is a non-zero partial isometry;

2. For any finite path $\mu$, the operator $S_\mu^*$ is a non-zero partial isometry;

3. For any finite paths $\mu, v$ with $s(v) = s(\mu)$, the operator $S_\mu S_v^*$ is a non-zero partial isometry;

4. Any finite product of $P+S+S^*$ can be written as an element of the form $S_\mu P_v S_v^*$; if $F$ is fibered, any such product can be written as a finite sum of elements of the form $S_\mu S_v^*$.

Proof. The proof is mainly based on the fact that $S_\mu^* S_\mu = P_{s(\mu)}$ for all path $\mu$; this fact is easily proved by induction.

1. Since $S_\mu^* S_\mu = P_{s(\mu)}$, $S_\mu$ is a partial isometry (recall that $u$ is a partial isometry if and only if $u^* u$ is a projection). It is non-zero because $P_{s(\mu)}$ is non-zero.

2. This is a direct consequence of the previous point.

3. Let $W = A_1 \ldots A_k$ be a word in $1 (S + S^* + P)^*$, and suppose that $W \neq 0$. Now, each subword of the form $S_P P_v$ should be different from $0$, which is equivalent to the fact that $v = s(e)$; then $S_P P_v = S_e S_e^* S_e = S_e$ using the condition (CK1) and the fact that $S_e$ is a partial isometry. Dually, each subword of the form $P_v S_e^*$ can be rewritten as $S_e^*$. Similarly, subwords of the form $P_v S_e$ should be non-zero, which is equivalent to the fact that $v = t(e)$. We notice that $S_v S_e^* \leq P_v$ and therefore $P_v S_v S_e^* = S_v S_e^*$, and conclude that $P_v S_v = P_v S_v S_e^* S_e = S_v S_e^* S_e = S_v$. Dually, every subword of the form $S_v^* P_v$ can be rewritten as $S_v^*$. Using the rewrite rules just exposed, we conclude that if $W$ contains an element of $(S + S^*)^*$, we can suppose it does not contain any element of $P$. If $W$ contains only elements of $P$, then they are all equal (since these projections are pairwise disjoint and $W$ should be non-zero) and therefore $W$ can be rewritten as a single element $P_\nu$ of $P$. If $F$ is fibered, there is an edge $e$ in the graph such that $t(e) = v$, we can use (CK2) to write $W$ as a sum of elements in $S^* (S^*)^*$.

Now, suppose that $W$ is a word in $(S + S^*)^*$ that is non-zero. Then we will explain how $W$ can be rewritten as a word in $S^* (S^*)^*$. If $W$ is already of the form $S_\mu^* S_\mu$, we have nothing to do. Otherwise, let us write $W$ as $S_\theta S_\mu^* S_v S_\xi^* W$ ($\theta$ and $W$ may be empty, $\xi$ may be empty only when $W$ is). We first claim that $S_\mu^* S_\mu$ is non-zero if and only if either there exists a path $\rho$, possibly empty, such that $v = \rho \cdot \mu$, or there exists a path $\rho$ such that $\mu = \rho \cdot v$. In the first case, we can write $S_\mu^* S_\mu$ as $S_\mu^* S_\mu S_\rho = P_{s(\rho)} S_\rho$ and then we have $W = S_\theta S_\mu^* S_\mu S_\rho W$. In the second case, $S_\mu^* S_\mu$ can be rewritten as $S_\mu^* S_\mu S_\nu = S_\mu^* P_{s(\nu)} = S_\mu^*$ and $W$ can be written as $S_\theta S_\mu^* S_\mu S_\nu W$. Now, in both cases, we have reduced the length of $W$, and strictly decreased the number of alternations between words in $S^*$ and words in $(S^*)^*$. Thus, applying this process inductively, we are bound to end up with an expression of $W$ of the form $S_\mu$, $S_\mu^*$, or $S_\mu^* S_\nu$. All these being elements of $S^* (S^*)^*$, we have proved the wanted result.

To end the proof completely, we now prove the claim. Consider that $S_\mu^* S_\nu$ is non-zero, and write $\mu = \mu' \cdot e$ and $v = v' \cdot f$. We show that $S_\mu^* S_\nu = S_{\mu'}^* S_{v'}$; the assertion then follows by a simple induction. We have $S_{\mu'}^* S_{v'} = S_{\mu'}^* S_{v'} S_{v'}$ since

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1To avoid confusion with the adjoint ($\cdot)^*$, we denote the Kleene star by ($\cdot)^*$. 

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$S^*_{\mu}S_{\nu}$ is non-zero, $S^*_{\mu}S_{f}$ has to be non-zero and therefore $e = f$. Moreover, in this case $S^*_{\mu}S_{f} = P_{s(e)}$, a projection greater than $P_{t(v')}$. i.e. $P_{s(e)}P_{t(v')} = P_{t(v')} = S_{\nu}^*S_{\nu}$. Thus $P_{s(e)}S_{\nu} = S_{\nu}$ (since $S_{\nu} = S_{\nu}S_{\nu}^*S_{\nu}$ as a partial isometry), and therefore $S^*_{\mu}S_{\nu} = S_{\mu}^*S_{\nu}^*S_{f}S_{\nu'} = S_{\mu}^*P_{s(e)}S_{\nu'} = S_{\mu}^*S_{\nu'}$.

We obtain the following theorem as a consequence of the previous lemma, showing the existence of non-degenerate $E$-families on a separable Hilbert space.

§3.9 Theorem.
The $C^*$-algebra $C^*((P,S))$ is equal to $\text{span}\{S^*_{\mu}P_{v}S_{v} \mid s(v) = v = s(\mu)\}$. If the graph $F$ is fibered, then $C^*((P,S))$ is equal to $\text{span}\{S^*_{\mu}S_{\nu} \mid s(v) = s(\mu)\}$.

Now, different $E$-families may give rise to different $C^*$-algebras. We are therefore interested in the universal $E$-family. This is a special case of the previous section.

§3.10 Theorem.
Let $E$ be a directed graph. We denote by $C^*(E)$ the universal $C^*$-algebra $C^*((\emptyset, R))$ where $\emptyset = P + S$ and $R$ are the relations corresponding to (CK1) and (CK2), together with the relations $P_{v} = P_{v}^*, P_{v}P_{w} = 0$ and $S_{e} = S_{e}S_{e}^*S_{e}$ for all $P_{v}, P_{w} \in P$ and all $S_{e} \in S$ respectively.

K-theory

Examples

4 Shift Spaces