Graphs of Interaction

Thomas Seiller (IML - Marseille)

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What are graphs of interaction

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2. Yields both a denotational semantic and a notion of truth.
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1. A localized semantic (for MLL).
2. Yields both a denotational semantic and a notion of truth.
3. It is inspired from the latest version of geometry of interaction GoI5 (Girard), and turns out to be a combinatorial version of it.
1. Graphs and loops
   - Plugging and Execution
   - Adjunction

2. Graphs of Interaction
   - Probabilistic GoI
   - Geometry of Interaction 5

3. Generalizations
   - Quantum GoI
   - Additives
Plugging corresponds to the recognition of a cut.
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**Definition**

If $G = (V, E)$ and $H = (V', F)$ are two graphs, we define $G □ H$ as the graph $G △ H = (V △ V', E △ F)$ together with a coloring function $\delta$ such that

$$\delta(e) = \begin{cases} 0 & \text{if } e \in E \\ 1 & \text{if } e \in F \end{cases}$$
We wish to define the execution in the same way it was defined on the first version of GoI with permutations (Multiplicatives).
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![Graph Diagram]

1 2 3 4

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**Definition**

We define the execution $F :: G$ between two graphs $F = (V, E)$ et $G = (V', F)$ as

$$F :: H = (V \Delta V', Path^{alt}(F □ G))$$

where $Path^{alt}(F □ G)$ is the set of alternating paths in $F □ G$. 
Moreover, our graphs are weighted, so we define the weight of a path as the product of the edges it is composed of.
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\[ \frac{\lambda \mu}{4} \]
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Counting loops

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Let $G_1 = (V_1, F_1)$ and $G_2 = (V_2, F_2)$ be two graphs, with $V \cap V' = \emptyset$. We define $G_1 \cup G_2$ as the graph $(V_1 \cup V_2, F_1 \uplus F_2)$. 
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**Proposition**

Let $F = (V_1 \cup V_2, E)$ be a graph. Denoting $\text{cycl}(F, G)$ the number of alternating loops between $F$ and $G$, we get:

$$\text{cycl}(F, G_1 \cup G_2) = \text{cycl}(F, G_1) + \text{cycl}(F :: G_1, G_2)$$
Counting loops

We will associate to a given couple of directed weighted graphs a number which is linked to the number of (alternating) loops. In fact, we will just sum the weights of the loops, normalized by their length.

\[ \sum_{\pi \in \text{Path}_{x,x}(F,G)} \frac{\omega_{F \Box G}(\pi)}{\log(\pi)} \]

This can be rewritten, by quotienting the set of loops by their starting point, as

\[ \sum_{\pi \in \text{AltLoops}(F,G)} \frac{\omega_{F \Box G}(\pi)}{\text{pow}(\pi)} \]

where \( \text{pow}(\pi) \) is the greatest integer \( k \) such that there exists a loop \( \rho \) satisfying \( \pi = \rho^k \).
Loops

Definition

We define a cycle as a loop $\pi$ such that $\text{pow}(\pi) = 1$. We write $Cycl(F, G)$ the set of alternating cycles in $F \boxdot G$.

Definition

$$\ll F, G \gg = \sum_{\pi \in Cycl(F, G)} \sum_{k \geq 1} \frac{\omega(\pi)^k}{k} = \sum_{\pi \in Cycl(F, G)} -\log(1 - \omega(\pi))$$
Proposition (Adjunction)

\[ \langle F, G_1 \cup G_2 \rangle = \langle F, G_1 \rangle + \langle F :: G_1, G_2 \rangle \]

Remark

If weights are in \( \{0, 1\} \), the third term \( \langle F, G_1 \rangle \) of the adjunction is either equal to 0 or to \( \infty \).
Graph reduction

Definition

We define the operation \( \hat{\cdot} \) that associates, to each weighted graph \( G \), a simple weighted graph by replacing the set \( E_{v,w} \) of edges from \( v \) to \( w \) by a single edge which weight is equal to \( \sum_{x \in E_{v,w}} \omega(x) \).
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Theorem
$\langle\langle F, G \rangle \rangle = \langle\langle F, \hat{G} \rangle \rangle$
Projects

Definition (Projects — Proofs)

A *project* is a couple \( \alpha = (a, A) \) where \( a \in \mathbb{R}_+ \cup \{\infty\} \) is the *wager* and \( A \) is a directed weighted graph (with weights in \([0, 1]\)). The finite set \( V_A \) of vertices of \( A \) will be called the *carrier* of \( \alpha \).

Thomas Seiller (IML - Marseille) Graphs of Interaction
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Let $\alpha$ and $\beta$ be two projects. We define

$$\ll \alpha, \beta \gg = a + b + \ll A, B \gg$$
Definition (Duality (Orthogonality))

Two projects $a$ and $b$ of same carrier are said to be *polar* when $\langle a, b \rangle \neq 0, \infty$. 
Conducts

**Definition (Duality (Orthogonality))**

Two projects $a$ and $b$ of same carrier are said to be polar when $\langle a, b \rangle \neq 0, \infty$.

**Definition (Conducts — Formulas)**

A conduct is a set of projects (with same carrier) equal to its bi-polar.
Connectives on projects

Definition
Let $a, b$ be two projects of disjoint carriers, we define

$$a \otimes b = (\ll a, b \gg, A \cup B)$$
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Definition (Cut)

Let $f$ and $g$ be two projects of respective carriers $V \cup V'$ and $V' \cup V''$. We define the cut between $f$ and $g$ as

$$f :: g = (\ll f, g \gg, F :: G)$$
Connectives on Conducts

**Definition**

Let $A, B$ be two conducts of disjoint carriers, we define

$$A \otimes B = \{a \otimes b \mid a \in A, b \in B\}$$
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Definition (Linear Implication)

Let $A, B$ be two conducts of disjoint carrier, we define

$$A \multimap B = \{f \mid \forall a \in A, f :: a \in B\}$$
Duality of the connectives

**Proposition**

\[ A \circ B = (A \otimes B^\perp)^\perp \]

**Proof.**

\[
\langle f, a \otimes b \rangle = f + (a + b) + \langle F, A \cup B \rangle \\
= f + a + b + \langle F, A \rangle + \langle F :: A, B \rangle \\
= (a + f + \langle F, A \rangle) + b + \langle F :: A, B \rangle \\
= \langle f :: a, b \rangle
\]
Definition
We define the category $\mathsf{Graph}_{MLL}$

$$\text{Obj} = \{ A \mid A = A \downarrow \downarrow \text{ of carrier } X_A \in \mathcal{P}_f(\mathbb{N}) \}$$

$$\text{Mor}[A, B] = \{ f \in \psi_0(A) \rightarrow \psi_1(B) \}$$

Proposition
We define functors from the connectives $\otimes$, $\rightarrow$ (with delocalizations) and the negation. With these definitions, the category $\mathsf{Graph}_{MLL}$ is a $\ast$-autonomous category.
Definition (Successful project)

A project $a = (0, A)$ is successful when $\hat{A}$ is symmetric and satisfies $\hat{A}^3 = \hat{A}$, $Tr(\hat{A}) = 0$. 
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Definition (Truth)

A conduct $A$ is true when it contains a successful project.
Proposition

The conducts $\mathbf{A}$ and $\mathbf{A} \perp$ can’t both be true.
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Proposition

If $f \in A \rightarrow B$ and $g \in B \rightarrow C$ are successful, then $f : g \in A \rightarrow C$ is successful.
Proposition

*The conducts \( A \) and \( A \perp \) can’t both be true.*

Proposition

*If \( f \in A \rightarrow B \) and \( g \in B \rightarrow C \) are successful, then \( f :: g \in A \rightarrow C \) is successful.*

Proposition (Weak internal completeness)

*If \( f \in A \otimes B \) is successful, there exists successful projects \( a \in A \) and \( b \in B \) such that \( f = a \otimes b \).*
We restrict ourselves to simple graphs $G$ such that the adjacency matrix of $G$ has norm $\leq 1$ and is hermitian (i.e. if $(v, w) \in E$, then $(w, v) \in E$ and $\omega(w) = \omega(v)$).
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We define an embedding $\Psi$ associating to a graph an operator in the hyperfinite factor, and extend it to projects.
From graphs to operators

- We restrict ourselves to simple graphs $G$ such that the adjacency matrix of $G$ has norm $\leq 1$ and is hermitian (i.e. if $(v, w) \in E$, then $(w, v) \in E$ and $\omega(w) = \omega(v)$).
- We define an embedding $\Psi$ associating to a graph an operator in the hyperfinite factor, and extend it to projects.
- We have:

$$
\begin{align*}
\text{Graphs} & \quad \text{Gol5} \\
\mathbf{a} \perp \mathbf{b} & \iff \Psi(\mathbf{a}) \perp \Psi(\mathbf{b}) \\
\Psi(\mathbf{a} :: \mathbf{b}) & = \Psi(\mathbf{a}) :: \Psi(\mathbf{b}) \\
\ll \mathbf{a}, \mathbf{b} \gg & = \ll (\mathbf{a}), (\mathbf{b}) \gg \\
\mathbf{a} \text{ is successful} & \Rightarrow \Psi(\mathbf{a}) \text{ is successful}
\end{align*}
$$
Complex weights

We extend our weights, and consider graphs weighted by complex numbers of module $\leq 1$. 
Complex weights

- We extend our weights, and consider graphs weighted by complex numbers of module $\leq 1$.
- Everything works nicely except the notion of truth which does not compose.
Example

Graphs and loops
Graphs of Interaction
Generalizations
Quantum GoI
Additives

-1

1 2 3 4 5 6
This is because we get "successful" projects that operate on other bases. For instance, the graph whose adjacency matrix is

\[
\begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}
\]

is written as a simple transposition in a well-chosen basis.
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$$\begin{pmatrix}
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is written as a simple transposition in a well-chosen basis. The solution: a subjective notion of truth, i.e. a notion of truth that depends on a basis.
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Additives

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1. Expand the range of weights to manage slices. We could for instance introduce boolean variables as in additive proof nets.
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2. We can consider *stratified* graphs, i.e. graphs with multiple sets of edges. All the definitions we had for the multiplicative work, we just have to define the operations slice by slice. The advantage: two notions of union arise naturally, one being the $\otimes$ and the other being the $\&$. 